

The rigged Hilbert space approach to the Gamow states

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Abstract

We use the resonances of the spherical shell potential to present a thorough description of the Gamow (quasinormal) states within the rigged Hilbert space. It will be concluded that the natural setting for the Gamow states is a rigged Hilbert space whose test functions fall off at infinity faster than Gaussians.

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I. INTRODUCTION

Resonances are intrinsic properties of a quantum system, and they describe the system's preferred ways of decaying. The experimental fingerprints of a resonance are either a sharp peak in the cross section or the exponential decay of the probability to find the unstable particle. The sharp peaks in the cross section are characterized by the energy E_R at which they occur and by their width Γ_R . Decay is characterized by the energy E_R of the particle and by its lifetime τ_R .

The Gamow states are the wave functions of resonances, and they are eigenvectors of the Hamiltonian with a complex eigenvalue. The real part of the complex eigenvalue is associated with the energy of the resonance, and the imaginary part is associated with the width. The time evolution of the Gamow eigenfunctions abides by the exponential decay law.

The Gamow states are able to describe both sharp peaks in the cross section and decay, in accordance with the phenomenological perception that resonances and unstable particles are two sides of the same phenomenon. As well, when the (complex) resonance energy tends to a (real) bound-state energy, the Gamow eigenfunction becomes a bound state, in accordance with the phenomenological perception that unstable states are only quantitatively, not qualitatively, different from bound states, the only difference being that unstable states have a non-zero width, whereas the width of stable states is zero.

In a way, the Gamow states complete the so-called Heisenberg program, according to which spectral lines, widths and lifetimes are all observable quantities, and quantum mechanics should be able to predict them.

Gamow introduced the energy eigenfunction with complex eigenvalue in his paper on α -decay of atomic nuclei [1], and its properties and applications have been considered by many authors, see for example [2–86] and references therein. A pedestrian introduction to these states can be found in [16, 30, 35]. Gamow's treatment does not fit within the Hilbert space though, because self-adjoint operators on a Hilbert space can only have real eigenvalues. Recall however that Dirac's bra-ket formalism does not fit within the Hilbert space but rather within the rigged Hilbert space. Similarly, the rigged Hilbert space mathematics asserts the legitimacy of Gamow's proposition. In the rigged Hilbert space language, the Gamow states are eigenvectors of the dual extension of the self-adjoint Hamiltonian. Such

extension can surely have complex eigenvalues [87].

A rigged Hilbert space (also called a Gelfand triplet) is a triad of spaces

$$\Phi \subset \mathcal{H} \subset \Phi^\times \quad (1.1)$$

such that \mathcal{H} is a Hilbert space, Φ is a dense subspace of \mathcal{H} , and Φ^\times is the anti-dual space of Φ . The space Φ has a topology that is finer than the topology inherited from \mathcal{H} . The space Φ^\times contains the continuous, antilinear functionals over Φ . Associated with the rigged Hilbert space (1.1), there is always another rigged Hilbert space,

$$\Phi \subset \mathcal{H} \subset \Phi', \quad (1.2)$$

where Φ' is called the dual space of Φ and contains the continuous, linear functionals over Φ . Since the space Φ^\times is bigger than $\mathcal{H}^\times \equiv \mathcal{H}$, and since Φ' is also bigger than $\mathcal{H}' \equiv \mathcal{H}$, some physically meaningful states that find no accommodation in \mathcal{H} will find accommodation in Φ^\times and Φ' . For example, the eigensolutions of the time-independent Schrödinger equation associated with either the scattering energies (the Lippmann-Schwinger kets $|E^\pm\rangle$) or with the resonant energies (the Gamow kets $|z_R\rangle$) find accommodation in Φ^\times , whereas the bras $\langle^\pm E|$ and $\langle z_R|$ find accommodation in Φ' .

The present paper is devoted to show how the rigged Hilbert space is able to accommodate the Gamow states. Throughout the paper, rather than working in a general setting, we will use the example of the spherical shell potential,

$$V(\mathbf{x}) = V(r) = \begin{cases} 0 & 0 < r < a \\ V_0 & a < r < b \\ 0 & b < r < \infty, \end{cases} \quad (1.3)$$

and restrict ourselves to the s partial wave. However, as explained in Appendix A of Ref. [63], the result is valid for any partial wave and for spherically symmetric potentials that fall off faster than exponentials.

For the potential (1.3), expressions such as those for the Gamow eigenfunctions and the S matrix depend on the square root of the energy rather than on the energy itself. It is, therefore, easier to do calculations with the wave number k ,

$$k = \sqrt{\frac{2m}{\hbar^2} E}, \quad (1.4)$$

rather than with the energy E . However, we will write most results in terms of the energy, because they tend to be simpler than in terms of the wave number. Also, when the energy and the wave number become complex, we will denote them by z and q ,

$$q = \sqrt{\frac{2m}{\hbar^2}} z , \quad (1.5)$$

and when they correspond to a resonance R , we will denote them by z_R and k_R ,

$$k_R = \sqrt{\frac{2m}{\hbar^2}} z_R . \quad (1.6)$$

We will re-write most expressions in Dirac's bra-ket notation, because of its simplicity, clarity and beauty.

Wave functions in the position representation, denoted by φ , and Gamow bras and kets, denoted by $\langle z_R|$ and $|z_R\rangle$, will have sometimes a superscript $+$ or $-$ attached to them, and it is important to understand what this superscript means. Let us suppose that φ is a Gaussian wave packet in the position representation. Such Gaussian could be either an “in” state, in which case we denote it by φ^+ , or an “out” state, in which case we denote it by φ^- . When we write φ^+ , the Gaussian will be expanded by the “in” Lippmann-Schwinger bras and kets, and its energy representation will always be the one associated with the “in” bras and kets. When we write φ^- , the Gaussian will be expanded by the “out” Lippmann-Schwinger bras and kets, and its energy representation will always be the one associated with the “out” bras and kets. Thus, the superscripts \pm are sort of “phase-space” labels, since they tell us which energy representation we are using, even though we may be working in the position representation. Physically, the superscripts \pm are a reminder of whether we have imposed the “in” or the “out” boundary conditions on the Gaussian packet. For the Gamow bras and kets, the meaning of the superscripts \pm is analogous.

Of all the previous attempts to describe the Gamow states within the rigged Hilbert space, our approach is closest to that of Bollini *et al.* [23, 24]. There are, however, two main differences between the present approach and that of Refs. [23, 24]. First, the test functions we are going to use fall off at infinity faster than Gaussians, whereas the test functions used in Refs. [23, 24] fall off at infinity faster than exponentials. We use test functions that fall off faster than Gaussians because they enable us to perform resonance expansions that include all the resonances of the system and that exhibit time asymmetry [88]. And second, we obtain the relation between the Breit-Wigner amplitude and the Gamow states by

transforming to the energy representation, whereas in Refs. [23, 24] such relation is obtained by transforming to the momentum representation.

In Sec. II, the Gamow states of the spherical shell potential will be constructed. The Gamow kets associated with resonances and anti-resonances will be defined as the solutions of homogeneous integral equations of the Lippmann-Schwinger type. We will solve these integral equations in the radial, position representation. In this representation, those integral equations are equivalent to the time-independent Schrödinger equation subject to a purely outgoing boundary condition (POBC). We will also obtain the “left” Gamow eigenfunctions and will comment on the analogy between bound and resonance states.

In Sec. III, we will apply the theory of distributions to construct the Gamow bras and kets, which in Sec. IV will be shown to be generalized eigenvectors of the Hamiltonian with complex eigenvalues. Also in Sec. IV, we will construct the rigged Hilbert spaces that accommodate the Gamow bras and kets.

Next, in Sec. V, we will obtain the energy representations of the Gamow bras and kets, and show that they can be written in terms of the complex delta function and the residue distribution.

In Sec. VI, we will let the energy run over the full real line in order to obtain the “energy representation” associated with the Breit-Wigner distribution. We will show how the complex delta function becomes the Breit-Wigner distribution in such “energy representation.” The results of Secs. V and VI will, in particular, provide a mathematical support for the results presented in Ref. [63].

The time evolution of the Gamow bras and kets will be calculated in Sec. VII. We will argue, although not fully prove, that the time evolution of a resonance ket is valid for positive times only, whereas the time evolution of an anti-resonance ket is valid for negative times only. Thus, the time evolution of resonances is given by (non-unitary) semigroups, which express the time asymmetry built into a decaying process. This time asymmetry seems to be what some authors such as Fonda *et al.* [89], Cohen-Tannoudji *et al.* [90], or Goldberger and Watson [91] have called the *irreversibility* of a decaying process.

For the sake of completeness, in Sec. VIII we will construct the resonant expansions and see how such expansions allow us to isolate each resonance’s contribution and to interpret the deviations from exponential decay [92].

In Sec. IX, we will present two analogies that help to understand the physical meaning

of the Gamow states. The first analogy is that between the resonance expansions, the Dirac expansions, and the classical Fourier expansions. The second analogy is that between the classical, quasinormal modes and the quantum mechanical resonances. We will also explain the physical reason why the Gamow eigenfunctions blow up exponentially at infinity.

II. THE GAMOW EIGENFUNCTIONS

The Gamow eigenfunctions are customarily defined as eigensolutions of the Schrödinger equation subject to the POBC. Although we could start the study of the Gamow states with that definition, we will follow instead a treatment parallel to that of the Lippmann-Schwinger equation [93–95]. We will define a Gamow state as the solution of an integral equation [11, 14] that has the POBC built into it. Needless to say, in the end the explicit solutions of that integral equation will be found by solving the Schrödinger equation subject to the POBC.

A. The integral equation of the Gamow states

The Gamow states are solutions of a homogeneous integral equation of the Lippmann-Schwinger type. If $z_R = E_R - i\Gamma_R/2$ denotes the complex energy associated with a resonance of energy E_R and width Γ_R , then the corresponding Gamow state $|z_R\rangle$ fulfills [11, 14]

$$|z_R\rangle = \frac{1}{z_R - H_0 + i0} V |z_R\rangle. \quad (2.1)$$

The $+i0$ in Eq. (2.1) means that we are working with the retarded free Green function, which has a purely outgoing boundary condition built into it. The retarded free Green function is analytically continued across the cut into the lower half plane of the second sheet of the Riemann surface, where the complex number z_R is located. Therefore, as pointed out in [14], Eq. (2.1) should be written as

$$|z_R\rangle = \lim_{E \rightarrow z_R} \frac{1}{E - H_0 + i0} V |E\rangle. \quad (2.2)$$

This notation intends to express that we first have to calculate the retarded free Green function $(E - H_0 + i0)^{-1}$ in the physical sheet, and then continue it across the cut into the lower half plane of the second sheet.

The integral equation (2.1) has the POBC built into it. To be more precise, in the position representation Eq. (2.1) is equivalent to the time-independent Schrödinger equation subject to the condition that far away from the potential region, the solution behave as a purely outgoing wave.

As is well known, to each resonance energy z_R there corresponds an anti-resonance energy z_R^* that lies in the upper half plane of the second sheet. The integral equation satisfied by the anti-resonance state $|z_R^*\rangle$ reads as

$$|z_R^*\rangle = \frac{1}{z_R^* - H_0 - i0} V |z_R^*\rangle = \lim_{E \rightarrow z_R^*} \frac{1}{E - H_0 - i0} V |E\rangle. \quad (2.3)$$

In contrast to Eq. (2.1), Eq. (2.3) has a purely *incoming* boundary condition built into it. That is, in the position representation, Eq. (2.3) is equivalent to the time-independent Schrödinger equation subject to the condition that far away from the potential region, the solution behave as a purely incoming wave.

B. The Gamow states in the position representation

In the radial, position representation, Eqs. (2.1) and (2.3) become

$$\langle r | z_R \rangle = \langle r | \frac{1}{z_R - H_0 + i0} V | z_R \rangle = \lim_{E \rightarrow z_R} \langle r | \frac{1}{E - H_0 + i0} V | E \rangle, \quad (2.4)$$

$$\langle r | z_R^* \rangle = \langle r | \frac{1}{z_R^* - H_0 - i0} V | z_R^* \rangle = \lim_{E \rightarrow z_R^*} \langle r | \frac{1}{E - H_0 - i0} V | E \rangle. \quad (2.5)$$

In [14], these integral equations are written as

$$u(r; z_R) = \lim_{E \rightarrow z_R} \int_0^\infty G_0^+(r, s; E) V(s) u(s; E) ds, \quad (2.6)$$

$$u(r; z_R^*) = \lim_{E \rightarrow z_R^*} \int_0^\infty G_0^-(r, s; E) V(s) u(s; E) ds, \quad (2.7)$$

where

$$u(r; z_R) = \langle r | z_R \rangle. \quad (2.8)$$

In order to obtain the explicit expressions of the Gamow eigenfunctions, instead of solving the integral equations (2.6) and (2.7), we solve the equivalent Schrödinger differential equation

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) \right) u(r; z_R) = z_R u(r; z_R), \quad (2.9)$$

subject to the boundary conditions built into those integral equations,

$$u(0; z_R) = 0, \quad (2.10)$$

$$u(r; z_R) \text{ is continuous at } r = a, b, \quad (2.11)$$

$$\frac{d}{dr}u(r; z_R) \text{ is continuous at } r = a, b, \quad (2.12)$$

$$u(r; z_R) \sim e^{ik_R r} \text{ as } r \rightarrow \infty, \quad (2.13)$$

where condition (2.13) is the POBC. In Eqs. (2.9)-(2.13), $u(r; z_R) \equiv \langle r | z_R \rangle$ can denote either a resonance or an anti-resonance state.

For the spherical shell potential (1.3), the only possible eigenvalues of Eq. (2.9) subject to (2.10)–(2.13) are the solutions of the following transcendental equation:

$$\mathcal{J}_+(z_R) = 0, \quad (2.14)$$

where \mathcal{J}_+ is the Jost function, see, e.g., Refs. [35, 93]. The solutions of this equation come as a denumerable number of complex conjugate pairs z_n, z_n^* . The number $z_n = E_n - i\Gamma_n/2$ is the n th resonance energy. The number $z_n^* = E_n + i\Gamma_n/2$ is the n th anti-resonance energy. The corresponding resonance and anti-resonance wave numbers are given by

$$k_n = \sqrt{\frac{2m}{\hbar^2} z_n}, \quad -k_n^* = \sqrt{\frac{2m}{\hbar^2} z_n^*}, \quad n = 1, 2, \dots, \quad (2.15)$$

which belong, respectively, to the fourth and third quadrants of the k -plane. For the potential (1.3), the resonance poles are simple (see [31] for an example of a potential that produces double poles).

In terms of the wave number k_n , the n th Gamow eigensolution reads

$$u(r; z_n) = u(r; k_n) = N_n \begin{cases} \frac{1}{\mathcal{J}_3(k_n)} \sin(k_n r) & 0 < r < a \\ \frac{\mathcal{J}_1(k_n)}{\mathcal{J}_3(k_n)} e^{iQ_n r} + \frac{\mathcal{J}_2(k_n)}{\mathcal{J}_3(k_n)} e^{-iQ_n r} & a < r < b \\ e^{ik_n r} & b < r < \infty, \end{cases} \quad (2.16)$$

where

$$Q_n = \sqrt{\frac{2m}{\hbar^2} (z_n - V_0)}, \quad (2.17)$$

N_n is a normalization factor,

$$N_n^2 = i \operatorname{res} [S(q)]_{q=k_n}, \quad (2.18)$$

and \mathcal{J}_1 – \mathcal{J}_3 are coefficients whose expressions follow from the matching conditions (2.11) and (2.12). The Gamow eigensolution associated with the n th anti-resonance pole reads

$$u(r; z_n^*) = u(r; -k_n^*) = M_n \begin{cases} \frac{1}{\mathcal{J}_3(-k_n^*)} \sin(-k_n^* r) & 0 < r < a \\ \frac{\mathcal{J}_1(-k_n^*)}{\mathcal{J}_3(-k_n^*)} e^{-iQ_n^* r} + \frac{\mathcal{J}_2(-k_n^*)}{\mathcal{J}_3(-k_n^*)} e^{iQ_n^* r} & a < r < b \\ e^{-ik_n^* r} & b < r < \infty, \end{cases} \quad (2.19)$$

where M_n is a normalization factor,

$$M_n^2 = \text{i res} [S(q)]_{q=-k_n^*} = (N_n^2)^*, \quad (2.20)$$

and where

$$-Q_n^* = \sqrt{\frac{2m}{\hbar^2}(z_n^* - V_0)}. \quad (2.21)$$

For the sake of brevity, we will label the anti-resonance wave numbers $-k_n^*$ and $-Q_n^*$, the energies z_n^* , the normalization factors M_n and the eigenfunctions $u(r; z_n^*)$ with a negative integer n as

$$k_n, Q_n, z_n, N_n, u(r; z_n) \quad n = -1, -2, \dots \quad (2.22)$$

This notation will enable us to write results that are true for both resonances and anti-resonances just once.

Since they are eigenfunctions of a linear differential operator, the Gamow eigenfunctions (2.16) and (2.19) are defined up to a normalization factor. The normalization we have adopted was introduced by Zeldovich [6], who used a Gaussian regulator to damp the exponential blowup of the Gamow eigenfunctions and obtain a meaningful normalization:

$$\lim_{\mu \rightarrow 0} \int_0^\infty dr e^{-\mu r^2} [u(r; z_n)]^2 = 1, \quad (2.23)$$

where $n = \pm 1, \pm 2, \dots$. Zeldovich's normalization has (at least) three advantages. First, it generalizes the normalization of bound states; second, the residue of the propagator at the resonance energy factors out as a product of two Gamow eigenfunctions, see Eq. (2.29) below; and third, Zeldovich's normalization makes $u(r; z_n)$ have dimensions of $1/\sqrt{\text{length}}$, so $|u(r; z_n)|^2$ has dimensions of a radial probability density, just like any normalized wave function in the position representation.

It is worthwhile noting that the expressions for the delta-normalized Lippmann-Schwinger eigenfunctions are different when expressed in terms of k from when expressed in terms of

E [94, 95]. However, similarly to bound states, the expressions for the normalized Gamow eigenfunctions are the same when expressed in terms of k_R as when expressed in terms of z_R , see Eqs. (2.16) and (2.19).

C. The “left” Gamow eigenfunctions

After having obtained the “right” Gamow eigenfunctions, which will be associated with the Gamow kets, it is easy to obtain the “left” Gamow eigenfunctions, which will be associated with the Gamow bras.

The “left” Gamow eigenfunctions can be obtained by complex Hermitian conjugation of the “right” Gamow eigenfunctions [97], or by analytic continuation of the “left” Lippmann-Schwinger eigenfunctions [95, 96]. The resulting “left” Gamow eigenfunction associated with the resonance (or anti-resonance) energy z_n is given by

$$\langle z_n | r \rangle = [u(r; z_n^*)]^*, \quad n = \pm 1, \pm 2, \dots \quad (2.24)$$

Thus, contrary to naive expectations, the “left” Gamow eigenfunction is not just the complex conjugate of the “right” eigenfunction, but the complex conjugated eigenfunction evaluated at the complex conjugated energy. Note that this procedure to obtain the “left” from the “right” eigenfunctions generalizes the procedure to obtain the “left” from the “right” eigenfunctions of both the bound and the scattering eigenfunctions. Because the Gamow eigenfunctions satisfy

$$[u(r; z_n^*)]^* = u(r; z_n), \quad n = \pm 1, \pm 2, \dots, \quad (2.25)$$

the “left” and the “right” Gamow eigenfunctions are actually the same eigenfunction,

$$\langle z_n | r \rangle = [u(r; z_n^*)]^* = u(r; z_n) = \langle r | z_n \rangle, \quad n = \pm 1, \pm 2, \dots \quad (2.26)$$

In terms of the wave number, Eq. (2.26) reads as

$$\langle k_n | r \rangle = [u(r; -k_n^*)]^* = u(r; k_n) = \langle r | k_n \rangle, \quad n = \pm 1, \pm 2, \dots \quad (2.27)$$

Note that Eq. (2.25) is a symmetry of the Gamow eigenfunctions, and it is such symmetry what in the end makes the “left” eigenfunction be equal to the “right” one. Note also that such symmetry does in general not hold when we change the normalization of the Gamow eigenfunctions—yet another reason to choose Zeldovich’s normalization.

Equation (2.26) makes it clear why Zeldovich's normalization for the Gamow states is written as in (2.23) rather than as

$$\lim_{\mu \rightarrow 0} \int_0^\infty dr e^{-\mu r^2} |u(r; z_n)|^2 = 1. \quad (2.28)$$

Also, Eq. (2.26) can be used to show that at a resonance (or anti-resonance) pole, the residue of the Green function is given by

$$\text{res}[G(r, s; z)]_{z=z_n} = \frac{\hbar^2}{m} k_n \text{res}[G(r, s; q)]_{q=k_n} = u(r; k_n) u(s; k_n), \quad n = \pm 1, \pm 2, \dots, \quad (2.29)$$

which in bra-ket notation becomes

$$\text{res}[\langle r | \frac{1}{z - H} | s \rangle]_{z=z_n} = \langle r | z_n \rangle \langle z_n | s \rangle, \quad n = \pm 1, \pm 2, \dots \quad (2.30)$$

Note that this factorization could have been used to define the above normalization of the Gamow states and to show that the “left” Gamow eigenfunction $\langle z_n | s \rangle$ is the same as the “right” Gamow eigenfunction $\langle s | z_n \rangle$.

D. Bound states

For the sake of simplicity in the expressions, we have chosen a potential that doesn't bind bound states. We would nevertheless like to briefly comment on what happens when bound states appear.

The bound states satisfy the same integral equation as the resonance states, and therefore they automatically follow from the Schrödinger equation subject to the POBC along with resonances. Thus, the eigenfunction $u(r; z_R)$ becomes a bound state when we substitute the complex resonance energy z_R by a real bound-state energy E_B . In addition, Zeldovich's normalization for the Gamow eigenfunctions reduces to the standard normalization of bound states when we substitute z_R by E_B .

III. THE GAMOW BRAS AND KETS

The Gamow eigenfunctions $u(r; z_n)$ are obviously not square integrable, i.e., they do not belong to the Hilbert space $L^2([0, \infty), dr)$. Thus, like the Lippmann-Schwinger eigenfunctions [93–95], the Gamow eigenfunctions must be treated as distributions. By treating them as distributions, we will be able to generate the Gamow bras and kets.

According to the theory of distributions [98], the Gamow ket $|z_n\rangle$ associated with the eigenfunction $u(r; z_n)$ must be defined as [93–95]

$$\begin{aligned} |z_n\rangle : \Phi_{\text{exp}} &\longmapsto \mathbb{C} \\ \varphi &\longmapsto \langle \varphi | z_n \rangle := \int_0^\infty dr [\varphi(r)]^* u(r; z_n), \quad n = \pm 1, \pm 2, \dots \end{aligned} \quad (3.1)$$

The elements $\varphi(r)$ of Φ_{exp} are such that their “nice behavior” compensates the “bad behavior” of $u(r; z_n)$ so the integral (3.1) makes sense. The space Φ_{exp} will be constructed in Sec. IV. In the bra-ket notation, definition (3.1) becomes

$$\langle \varphi | z_n \rangle = \int_0^\infty dr \langle \varphi | r \rangle \langle r | z_n \rangle. \quad (3.2)$$

Similarly, the Gamow bras associated with the resonance (or anti-resonance) energy z_n are defined as

$$\begin{aligned} \langle z_n | : \Phi_{\text{exp}} &\longmapsto \mathbb{C} \\ \varphi &\longmapsto \langle z_n | \varphi \rangle := \int_0^\infty dr \varphi(r) u(r; z_n), \quad n = \pm 1, \pm 2, \dots; \end{aligned} \quad (3.3)$$

that is,

$$\langle z_n | \varphi \rangle = \int_0^\infty dr \langle z_n | r \rangle \langle r | \varphi \rangle. \quad (3.4)$$

From the above definitions and from Eq. (2.26), it follows that the actions of the Gamow bras and kets are related by

$$\langle \varphi | z_n \rangle = \langle z_n^* | \varphi \rangle^*, \quad n = \pm 1, \pm 2, \dots \quad (3.5)$$

Since the Gamow eigenfunctions are the same when expressed in terms of the energy as when expressed in terms of the wave number, the Gamow bras and kets, unlike the delta-normalized Lippmann-Schwinger bras and kets, are the same when expressed in terms of the energy as when expressed in terms of the wave number:

$$|k_n\rangle = |z_n\rangle, \quad \langle k_n| = \langle z_n|, \quad n = \pm 1, \pm 2, \dots \quad (3.6)$$

IV. THE RIGGED HILBERT SPACES FOR THE GAMOW BRAS AND KETS

Likewise any bra or ket, the Gamow bras and kets are dealt with by means of the rigged Hilbert space rather than just by the Hilbert space. The rigged Hilbert space we will use is

very similar to, although not the same as the rigged Hilbert space of Refs. [23, 24]. We will denote the rigged Hilbert space for the bras by

$$\Phi_{\text{exp}} \subset L^2([0, \infty), dr) \subset \Phi'_{\text{exp}}, \quad (4.1)$$

and the one for the kets by

$$\Phi_{\text{exp}} \subset L^2([0, \infty), dr) \subset \Phi_{\text{exp}}^\times. \quad (4.2)$$

The procedure to construct the space of test functions Φ_{exp} has been explained in [93–95]. The most important property one has to look at is the “bad behavior” of the Gamow eigenfunctions. Such “bad behavior” must be compensated by the “nice behavior” of the elements of Φ_{exp} so the integrals (3.1)-(3.4) converge. Since the regular solution $\chi(r; q)$ of the Schrödinger equation is related to the Gamow eigenfunction by

$$\chi(r; k_n) = \frac{1}{2i} \frac{\mathcal{J}_-(k_n)}{N_n} u(r; k_n), \quad n = \pm 1, \pm 2, \dots, \quad (4.3)$$

and since by, for example, Eq. (12.6) in Ref. [99] the regular solution satisfies

$$|\chi(r; q)| \leq C \frac{|q| r}{1 + |q| r} e^{|\text{Im}(q)|r}, \quad q \in \mathbb{C}, \quad (4.4)$$

the “bad behavior” of the Gamow eigenfunctions is given by

$$|u(r; k_n)| \leq C \frac{|N_n|}{|\mathcal{J}_-(k_n)|} \frac{|k_n| r}{1 + |k_n| r} e^{|\text{Im}(k_n)|r}, \quad n = \pm 1, \pm 2, \dots. \quad (4.5)$$

Because the bound (4.4) is sharp [99], so is the bound (4.5). Thus, the Gamow eigenfunctions grow exponentially as r tends to infinity, and, in order for the integrals (3.1)-(3.4) to converge, the wave functions of Φ_{exp} must fall off at infinity sufficiently rapidly.

From Eq. (4.5), it is clear that the integrals in Eqs. (3.1)-(3.4) converge already for functions that fall off at infinity faster than any exponential [23, 24]. Thus, exponential falloff is the weakest falloff that we need to require from the wave functions of Φ_{exp} , see Refs. [23, 24]. However, we are going to impose a stronger, Gaussian falloff because it allows us to perform certain resonance expansions, as will be discussed in Sec. VIII.

Using the estimate (4.5), and following the procedure of [93–95] to construct spaces of test functions, one ends up finding that Φ_{exp} is given by

$$\Phi_{\text{exp}} = \{\varphi \in \mathcal{D} \mid \|\varphi\|_{m, m'} < \infty, \quad m, m' = 0, 1, 2, \dots\}, \quad (4.6)$$

where \mathcal{D} is the maximal invariant subspace of the Hamiltonian,

$$\mathcal{D} = \bigcap_{m=0}^{\infty} \mathcal{D}(H^m), \quad (4.7)$$

and $\|\cdot\|_{m,m'}$ is given by

$$\|\varphi\|_{m,m'} := \sqrt{\int_0^\infty dr \left| \frac{mr}{1+mr} e^{mr^2/2} (1+H)^{m'} \varphi(r) \right|^2}, \quad m, m' = 0, 1, 2, \dots \quad (4.8)$$

Hence, Φ_{exp} is just the space of square integrable functions which belong to the maximal invariant subspace of H and for which the quantities (4.8) are finite. In particular, because $\varphi(r)$ satisfies the estimates (4.8), $\varphi(r)$ falls off at infinity faster than e^{-r^2} , that is, its tails fall off faster than Gaussians.

Note that we have arrived at the same space of test functions as the one for the analytically continued Lippmann-Schwinger bras and kets [95], since also in that case we have to tame real exponentials.

Once we have constructed the space Φ_{exp} , we can construct its dual Φ'_{exp} and antidual Φ_{exp}^\times spaces as the spaces of, respectively, linear and antilinear continuous functionals over Φ_{exp} , and therewith the rigged Hilbert spaces (4.1) and (4.2). The Gamow bras and kets are, respectively, linear and antilinear continuous functionals over Φ_{exp} . As well, they are (generalized) eigenvectors of the Hamiltonian.

The following proposition, whose proof follows exactly the same steps as the proof of Proposition 2 in [95], encapsulates the results of this section:

Proposition 1. *The triplets of spaces (4.1) and (4.2) are rigged Hilbert spaces, and they satisfy all the requirements to accommodate the Gamow bras and kets. More specifically,*

- (i) *The $\|\cdot\|_{m,m'}$ are norms, and they define a countably normed topology, i.e., a meaning of sequence convergence.*
- (ii) *The space Φ_{exp} is dense in $L^2([0, \infty), dr)$.*
- (iii) *The space Φ_{exp} is invariant under the action of the Hamiltonian, and H is Φ_{exp} -continuous.*
- (iv) *The kets $|z_n\rangle$ are continuous, antilinear functionals over Φ_{exp} , i.e., $|z_n\rangle \in \Phi_{\text{exp}}^\times$.*

(v) The kets $|z_n\rangle$ are generalized “right” eigenvectors of H with eigenvalue z_n :

$$H|z_n\rangle = z_n|z_n\rangle, \quad n = \pm 1, \pm 2, \dots; \quad (4.9)$$

that is,

$$\langle\varphi|H|z_n\rangle = z_n\langle\varphi|z_n\rangle, \quad \varphi \in \Phi_{\text{exp}}. \quad (4.10)$$

(vi) The bras $\langle z_n|$ are continuous, linear functionals over Φ_{exp} , i.e., $\langle z_n| \in \Phi'_{\text{exp}}$.

(vii) The bras $\langle z_n|$ are generalized “left” eigenvectors of H with eigenvalue z_n :

$$\langle z_n|H = z_n\langle z_n|, \quad n = \pm 1, \pm 2, \dots; \quad (4.11)$$

that is,

$$\langle z_n|H|\varphi\rangle = z_n\langle z_n|\varphi\rangle, \quad \varphi \in \Phi_{\text{exp}}. \quad (4.12)$$

Proposition 1 makes it clear, in particular, that there is a 1:1 correspondence between Gamow bras and kets.

Note that in terms of the wave number, the eigenequations (4.9) and (4.11) become

$$H|k_n\rangle = \frac{\hbar^2}{2m}k_n^2|k_n\rangle, \quad n = \pm 1, \pm 2, \dots, \quad (4.13)$$

$$\langle k_n|H = \frac{\hbar^2}{2m}k_n^2\langle k_n|, \quad n = \pm 1, \pm 2, \dots. \quad (4.14)$$

Note also that the bra eigenequation (4.11) is *not* given by

$$\langle z_n|H = z_n^*\langle z_n|, \quad (4.15)$$

as one may naively obtain by Hermitian conjugation of the ket eigenequation (4.9). The reason lies in that one has to use complex Hermitian conjugation to obtain the “left” from the “right” Gamow eigenfunction, see Eq. (2.26).

The normalization condition satisfied by the Gamow states is the following:

$$\langle z_n|z_{n'}\rangle = \delta_{n,n'}, \quad n, n' = \pm 1, \pm 2, \dots. \quad (4.16)$$

When $n = n'$, Eq. (4.16) follows from Zeldovich’s regularization (2.23). When $n \neq n'$, Eq. (4.16) can be proved in the same way as one proves the orthogonality of bound states:

$$\langle z_n|H|z_{n'}\rangle = z_n\langle z_n|z_{n'}\rangle = z_{n'}\langle z_n|z_{n'}\rangle, \quad (4.17)$$

where we have made use of the fact that $\langle z_n|$ and $|z_{n'}\rangle$ are eigenvectors of H with eigenvalue z_n and $z_{n'}$, respectively. From the second equality in (4.17), we obtain

$$(z_n - z_{n'})\langle z_n|z_{n'}\rangle = 0, \quad (4.18)$$

which yields the desired result, since $z_n \neq z_{n'}$ when $n \neq n'$. It should be noted however that, similar to the normalization of scattering states, the normalization condition (4.16) has only formal meaning and does not imply the use of a Hilbert-space scalar product. For example, the “scalar product” built on Eqs. (4.16) and (2.23) would not satisfy $(f, f) \geq 0$.

V. THE ENERGY REPRESENTATIONS OF RIGGED HILBERT SPACES AND OF THE GAMOW BRAS AND KETS

We turn now to obtain and characterize the energy representations of the rigged Hilbert spaces (4.1) and (4.2) and of the Gamow bras and kets. It is here where we will need to introduce the labels \pm in the notation for the wave functions and for the Gamow bras and kets.

A. The energy representations of the rigged Hilbert space

The “in” and the “out” energy representations of Φ_{exp} are readily obtained by means of the unitary operators U_{\pm} of [94]:

$$U_{\pm}\Phi_{\text{exp}} \equiv \hat{\Phi}_{\pm\text{exp}}, \quad (5.1)$$

which in turn yield the energy representations of the rigged Hilbert spaces (4.1) and (4.2):

$$\hat{\Phi}_{\pm\text{exp}} \subset L^2([0, \infty), dE) \subset \hat{\Phi}'_{\pm\text{exp}}, \quad (5.2)$$

$$\hat{\Phi}_{\pm\text{exp}} \subset L^2([0, \infty), dE) \subset \hat{\Phi}_{\pm\text{exp}}^{\times}. \quad (5.3)$$

The elements of $\hat{\Phi}_{\pm\text{exp}}$ will be denoted by $\hat{\varphi}^{\pm}(z) = U_{\pm}\varphi(z)$.

In [95], we characterized the analytic and growth properties of the wave functions in the wave number representations, $\hat{\varphi}^{\pm}(q)$, which are related to the wave functions in the energy representations as

$$\hat{\varphi}^{\pm}(z) = \sqrt{\frac{2m}{\hbar^2}} \frac{1}{2q} \hat{\varphi}^{\pm}(q). \quad (5.4)$$

Thus, the results of [95] also characterize the analytic and growth properties of $\widehat{\varphi}^\pm(z)$, and we will refer to [95] whenever we need to make use of any such properties.

As mentioned above, from now on we will add a label to the action of the Gamow states,

$$\langle \varphi^\pm | z_n^\pm \rangle, \quad \langle {}^\pm z_n | \varphi^\pm \rangle, \quad n = \pm 1, \pm 2, \dots \quad (5.5)$$

When we use the label $+$, it will mean that the energy representation is obtained through the operator U_+ , and when we use the label $-$, it will mean that the energy representation is obtained through the operator U_- .

B. The energy representations of the Gamow bras and kets

In order to obtain the energy representations of the Gamow bras and kets, we first need to define the linear complex delta functional at z :

$$\begin{aligned} \langle \widehat{\delta}_z | : \widehat{\Phi}_{\text{exp}} &\longmapsto \mathbb{C} \\ \widehat{\varphi} &\longmapsto \langle \widehat{\delta}_z | \widehat{\varphi} \rangle := \widehat{\varphi}(z), \end{aligned} \quad (5.6)$$

where $\widehat{\Phi}_{\text{exp}}$ may be either $\widehat{\Phi}_{+\text{exp}}$ or $\widehat{\Phi}_{-\text{exp}}$, and $\widehat{\varphi}$ may be either $\widehat{\varphi}^+$ or $\widehat{\varphi}^-$. Thus, the linear complex delta functional at z associates with each test function, the value of the test function at z . One can write (5.6) as an integral operator as

$$\langle \widehat{\delta}_z | \widehat{\varphi} \rangle = \int_0^\infty dE \delta(E - z) \widehat{\varphi}(E) = \widehat{\varphi}(z). \quad (5.7)$$

In this way, one can interpret the complex delta function $\delta(E - z)$ as the analytic continuation of the Dirac delta function $\delta(E - E')$.

The antilinear complex delta functional $|\widehat{\delta}_z\rangle$ at the complex number z can be defined in a similar way:

$$\begin{aligned} |\widehat{\delta}_z\rangle : \widehat{\Phi}_{\text{exp}} &\longmapsto \mathbb{C} \\ \widehat{\varphi} &\longmapsto \langle \widehat{\varphi} | \widehat{\delta}_z \rangle := [\widehat{\varphi}(z^*)]^*. \end{aligned} \quad (5.8)$$

We also need to define the linear and antilinear residue functionals at z :

$$\begin{aligned} \langle \widehat{\text{res}}_z | : \widehat{\Phi}_{\text{exp}} &\longmapsto \mathbb{C} \\ \widehat{\varphi} &\longmapsto \langle \widehat{\text{res}}_z | \widehat{\varphi} \rangle := \text{res}[\widehat{\varphi}(z)], \end{aligned} \quad (5.9)$$

$$\begin{aligned} |\widehat{\text{res}}_z\rangle : \widehat{\Phi}_{\text{exp}} &\longmapsto \mathbb{C} \\ \widehat{\varphi} &\longmapsto \langle \widehat{\varphi} | \widehat{\text{res}}_z \rangle := \text{res}[\widehat{\varphi}(z^*)]^*, \end{aligned} \quad (5.10)$$

where $\text{res}[\widehat{\varphi}(z)]$ stands for the residue of $\widehat{\varphi}$ at z . Both the complex delta functionals and the residue functionals at z are well defined when the test functions can be analytically continued into z , as is our case [95].

We will need also the following normalization factor:

$$\mathcal{N}_n^2 = \text{i res}[S(z)]_{z=z_n} = \text{i} \frac{\hbar^2}{2m} 2k_n \text{res}[S(q)]_{q=k_n} = \frac{\hbar^2}{2m} 2k_n N_n^2, \quad (5.11)$$

where N_n was used in Sec. II to normalize the Gamow eigenfunctions.

If we denote the energy representations of the Gamow bras and kets as

$$\langle^\pm \widehat{z}_n| \equiv \langle^\pm z_n| U_\pm, \quad (5.12)$$

$$|\widehat{z}_n^\pm\rangle \equiv U_\pm |z_n^\pm\rangle, \quad (5.13)$$

then the following proposition, whose proof can be found in Appendix A, holds:

Proposition 2. *For a resonance (or anti-resonance) of energy z_n , the “minus” (or “out”) energy representation of the Gamow bras and kets is given by*

$$\langle^- \widehat{z}_n| = -\frac{\sqrt{2\pi}}{\mathcal{N}_n} \langle \widehat{\text{res}}_{z_n}|, \quad n = \pm 1, \pm 2, \dots, \quad (5.14)$$

$$|\widehat{z}_n^-\rangle = \text{i}\sqrt{2\pi} \mathcal{N}_n |\widehat{\delta}_{z_n}\rangle, \quad n = \pm 1, \pm 2, \dots. \quad (5.15)$$

Their “plus” energy representation is given by

$$\langle^+ \widehat{z}_n| = \text{i}\sqrt{2\pi} \mathcal{N}_n \langle \widehat{\delta}_{z_n}|, \quad n = \pm 1, \pm 2, \dots, \quad (5.16)$$

$$|\widehat{z}_n^+\rangle = -\frac{\sqrt{2\pi}}{\mathcal{N}_n} |\widehat{\text{res}}_{z_n}\rangle, \quad n = \pm 1, \pm 2, \dots. \quad (5.17)$$

Proposition 2 shows, in particular, that the “plus” energy representation of a Gamow bra or ket is different from its “minus” energy representation, thereby showing that the labels \pm matter.

The complex delta functional and the residue functional can be written in more familiar terms as follows. By using the resolutions of the identity

$$I = \int_0^\infty \text{d}E |E^\pm\rangle \langle^\pm E|, \quad (5.18)$$

we can formally write the actions of $\langle^\pm \widehat{z}_n|$ as integral operators and obtain

$$\begin{aligned}\langle^\pm \widehat{z}_n|\widehat{\varphi}^\pm\rangle &= \langle^\pm z_n|\varphi^\pm\rangle \\ &= \int_0^\infty dE \langle^\pm z_n|E^\pm\rangle \langle^\pm E|\varphi^\pm\rangle \\ &= \int_0^\infty dE \langle^\pm z_n|E^\pm\rangle \widehat{\varphi}^\pm(E).\end{aligned}\tag{5.19}$$

Comparison of (5.19) with (5.14) and (5.16) shows that $\langle^- z_n|E^- \rangle$ is proportional to the residue distribution,

$$\langle^- z_n|E^- \rangle = -\frac{\sqrt{2\pi}}{\mathcal{N}_n} \text{res}[\cdot]_{z_n}, \quad E \geq 0, \quad n = \pm 1, \pm 2, \dots, \tag{5.20}$$

and that $\langle^+ z_n|E^+ \rangle$ is proportional to the complex delta function,

$$\langle^+ z_n|E^+ \rangle = i\sqrt{2\pi} \mathcal{N}_n \delta(E - z_n), \quad E \geq 0, \quad n = \pm 1, \pm 2, \dots. \tag{5.21}$$

Similarly, by using (5.18) we can formally write the actions of $|\widehat{z}_n^\pm\rangle$ as integral operators:

$$\begin{aligned}\langle\widehat{\varphi}^\pm|\widehat{z}_n^\pm\rangle &= \langle\varphi^\pm|z_n^\pm\rangle \\ &= \int_0^\infty dE \langle\varphi^\pm|E^\pm\rangle \langle^\pm E|z_n^\pm\rangle \\ &= \int_0^\infty dE [\varphi^\pm(E)]^* \langle^\pm E|z_n^\pm\rangle.\end{aligned}\tag{5.22}$$

By comparing (5.22) with (5.15) and (5.17), we deduce that $\langle^- E|z_n^- \rangle$ is proportional to the complex delta function

$$\langle^- E|z_n^- \rangle = i\sqrt{2\pi} \mathcal{N}_n \delta(E - z_n), \quad E \geq 0, \quad n = \pm 1, \pm 2, \dots, \tag{5.23}$$

and that $\langle^+ E|z_n^+ \rangle$ is proportional to the residue distribution,

$$\langle^+ E|z_n^+ \rangle = -\frac{\sqrt{2\pi}}{\mathcal{N}_n} \text{res}[\cdot]_{z_n}, \quad E \geq 0, \quad n = \pm 1, \pm 2, \dots. \tag{5.24}$$

It is important to realize that with a given test function, the complex delta function and the residue distribution at z_n associate, respectively, the value and the residue of the analytic continuation of the test function at z_n . This is why when those distributions act on $[\widehat{\varphi}^\pm(E)]^*$ as in Eq. (5.22), the final result is respectively $[\widehat{\varphi}^-(z_n^*)]^*$ and $\text{res}[\widehat{\varphi}^+(z_n^*)]^*$, rather than $[\widehat{\varphi}^-(z_n)]^*$ and $\text{res}[\widehat{\varphi}^+(z_n)]^*$, since the analytic continuation of $[\widehat{\varphi}^\pm(E)]^*$ is $[\widehat{\varphi}^\pm(z^*)]^*$ rather than $[\widehat{\varphi}^\pm(z)]^*$.

VI. THE $(-\infty, \infty)$ -“ENERGY” REPRESENTATION

The spectrum of our Hamiltonian is $[0, \infty)$. Hence, in Eqs. (5.21) and (5.23) the energy E runs over the positive real line. In this section, we are going to let E run over the full real line. In doing so, we can see what would happen if the spectrum of the Hamiltonian wasn't bounded from below.

It is important to keep in mind that in this section, we will need to treat resonances and anti-resonances separately. Also, strictly speaking, whenever we say that E runs over over the full real line $(-\infty, \infty)$, it will actually mean that in the case of resonances (anti-resonances), E runs infinitesimally below (above) the real axis of the second sheet of the Riemann surface.

In order to construct the $(-\infty, \infty)$ -“energy” representation, we will construct the transform \mathbb{A} that lets the energy vary over the full real line. The transform \mathbb{A} is modeled after the “ θ transform” of [19], and it allows us to connect the physical spectrum, which in our case coincides with $[0, \infty)$, with the support of the Breit-Wigner amplitude, which coincides with $(-\infty, \infty)$. Basically, \mathbb{A} takes a test function $\hat{\varphi}^\pm(E)$, $E \geq 0$, of $\hat{\Phi}_{\pm\text{exp}}$ into its analytic continuation over the full real line, $\hat{\varphi}^\pm(E)$, $E \in (-\infty, \infty)$. In order to distinguish when the energy runs over the physical spectrum from when it runs over the full real line, we will denote $\hat{\varphi}^\pm(E)$, $E \in (-\infty, \infty)$, by $\tilde{\varphi}^\pm(E)$ and thus will write

$$\mathbb{A}\hat{\varphi}^\pm \equiv \tilde{\varphi}^\pm, \quad (6.1)$$

and

$$\mathbb{A}\hat{\Phi}_{\pm\text{exp}} \equiv \tilde{\Phi}_{\pm\text{exp}} = \{\tilde{\varphi}^\pm(E) \mid E \in (-\infty, \infty)\}. \quad (6.2)$$

The following diagram shows how \mathbb{A} links the energy representation with the $(-\infty, \infty)$ -“energy” representation:

$$\begin{array}{ccc} \hat{\varphi}^\pm, \hat{\Phi}_{\pm\text{exp}} \subset L^2([0, \infty), dE) \subset \hat{\Phi}_{\pm\text{exp}}^\times & \text{energy representation} & \\ \downarrow \mathbb{A} & & \downarrow \mathbb{A} \\ \tilde{\varphi}^\pm, \tilde{\Phi}_{\pm\text{exp}} \subset L^2(\mathbb{R}, d_\alpha E) \subset \tilde{\Phi}_{\pm\text{exp}}^\times & (-\infty, \infty)\text{-“energy” repr.} & \end{array} \quad (6.3)$$

where $L^2(\mathbb{R}, d_\alpha E)$ is the following space:

$$L^2(\mathbb{R}, d_\alpha E) = \{\tilde{f} \mid \hat{f} \in L^2([0, \infty), dE), \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} dE |e^{-iE\alpha} \tilde{f}(E)|^2 < \infty\}. \quad (6.4)$$

In Eq. (6.4), the integral is assumed to be calculated in the second sheet, infinitesimally below (or above, in the case of anti-resonances) the real axis. The convergence factor $e^{-iE\alpha}$ (which becomes $e^{iE\alpha}$ in the case of anti-resonances) is needed because the analytic continuation of $\widehat{\varphi}^\pm(E)$ into the negative energies blows up exponentially [95]. Actually, if it wasn't needed, the spectrum would be the full real line. Nevertheless, the space $L^2(\mathbb{R}, d_\alpha E)$ is not crucial to our discussion.

It is important to understand that although we have denoted the functions $\widehat{\varphi}^\pm$ and $\widetilde{\varphi}^\pm = \mathbb{A}\widehat{\varphi}^\pm$ by a different symbol, they are indeed the *same* function. More precisely, they are different “pieces” of the same function. In particular, the value of their analytic continuation at a complex number z is the same,

$$\widetilde{\varphi}^\pm(z) = \widehat{\varphi}^\pm(z). \quad (6.5)$$

Obviously, the analytic continuation of their complex conjugates enjoys an analogous property,

$$[\widetilde{\varphi}^\pm(z^*)]^* = [\widehat{\varphi}^\pm(z^*)]^*. \quad (6.6)$$

We use different symbols for different “pieces” of the same function because the proof of the connection between the Breit-Wigner amplitude and the complex delta function becomes more transparent. For resonances, such connection is given by

$$\mathbb{A}|\widehat{z}_n\rangle = |\frac{1}{E - z_n}^-\rangle, \quad n = 1, 2, \dots, \quad (6.7)$$

where the ket $|\frac{1}{E - z_n}^-\rangle$ is associated with the Breit-Wigner amplitude as follows:

$$\begin{aligned} |\frac{1}{E - z_n}^-\rangle : \widetilde{\Phi}_{-\text{exp}} &\mapsto \mathbb{C} \\ \widetilde{\varphi}^- &\mapsto \langle \widetilde{\varphi}^- | \frac{1}{E - z_n}^- \rangle := \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} dE e^{-iE\alpha} \left(-\frac{\mathcal{N}_n}{\sqrt{2\pi}} \frac{1}{E - z_n} \right) [\widetilde{\varphi}^-(E)]^*. \end{aligned} \quad (6.8)$$

We will call this ket the Breit-Wigner ket. The integral in Eq. (6.8) is supposed to be calculated in the lower half plane of the second sheet, infinitesimally below the real axis. By the properties of $\widehat{\varphi}^-(z)$ in the lower half plane of the second sheet [95], the Breit-Wigner ket is a well defined antilinear functional. The proof of (6.7) is provided in Appendix A.

The combination of (6.7) with the results of Sec. V shows that the Gamow eigenfunction $u(r; z_n)$, the complex delta function (multiplied by a normalization factor) and the Breit-Wigner amplitude (multiplied by a normalization factor) are the same distribution in

different representations:

$$u(r; z_n) \leftrightarrow i\sqrt{2\pi}\mathcal{N}_n\delta(E - z_n), \quad E \in [0, \infty) \leftrightarrow -\frac{\mathcal{N}_n}{\sqrt{2\pi}}\frac{1}{E - z_n}, \quad E \in (-\infty, \infty) \quad (6.9)$$

posit. repr. energy repr. $(-\infty, \infty)$ -“energy” repr.

Physically, these links mean that the Gamow states yield a decay amplitude given by the complex delta function, and that such decay amplitude can be approximated by the Breit-Wigner amplitude when we can ignore the lower bound of the energy, i.e., when the resonance is so far from the threshold that we can safely assume that the energy runs over the full real line. However, because there is actually a lower bound for the energy, the decay amplitude is never exactly given by the Breit-Wigner amplitude. Mathematically, the reason lies in that \mathbb{A} is not unitary, which makes the energy representation be not equivalent to the $(-\infty, \infty)$ -“energy” representation.

One can also relate the “plus” Gamow bra with a Breit-Wigner bra:

$$\langle^+\widehat{z}_n|\mathbb{A} = \langle^+\frac{1}{E - z_n}|, \quad n = 1, 2, \dots, \quad (6.10)$$

where the Breit-Wigner bra is defined as

$$\begin{aligned} \langle^+\frac{1}{E - z_n}| : \widetilde{\Phi}_{+\text{exp}} &\mapsto \mathbb{C} \\ \widetilde{\varphi}^+ &\mapsto \langle^+\frac{1}{E - z_n}|\widetilde{\varphi}^+ \rangle := \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} dE e^{-iE\alpha} \left(-\frac{\mathcal{N}_n}{\sqrt{2\pi}}\frac{1}{E - z_n} \right) \widetilde{\varphi}^+(E). \end{aligned} \quad (6.11)$$

The proof of (6.10) is almost identical to the proof of (6.7).

For the anti-resonance energies, we obtain similar results to (6.7) and (6.10). The Gamow ket of an anti-resonance is related to a Breit-Wigner ket as

$$\mathbb{A}|\widehat{z}_n^- \rangle = |\frac{1}{E - z_n}^- \rangle, \quad n = -1, -2, \dots, \quad (6.12)$$

where now the ket $|\frac{1}{E - z_n}^- \rangle$ is associated with the Breit-Wigner amplitude as follows:

$$\begin{aligned} |\frac{1}{E - z_n}^- \rangle : \widetilde{\Phi}_{-\text{exp}} &\mapsto \mathbb{C} \\ \widetilde{\varphi}^- &\mapsto \langle \widetilde{\varphi}^- | \frac{1}{E - z_n}^- \rangle := \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} dE e^{iE\alpha} \left(\frac{\mathcal{N}_n}{\sqrt{2\pi}}\frac{1}{E - z_n} \right) [\widetilde{\varphi}^-(E)]^*. \end{aligned} \quad (6.13)$$

Similarly, the Gamow bra is associated with a Breit-Wigner bra as

$$\langle^+\widehat{z}_n|\mathbb{A} = \langle^+\frac{1}{E - z_n}|, \quad n = -1, -2, \dots, \quad (6.14)$$

where the Breit-Wigner bra is now defined as

$$\begin{aligned} \langle +\frac{1}{E-z_n} | : \tilde{\Phi}_{+\text{exp}} &\mapsto \mathbb{C} \\ \tilde{\varphi}^+ &\mapsto \langle +\frac{1}{E-z_n} | \tilde{\varphi}^+ \rangle := \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} dE e^{iE\alpha} \left(\frac{\mathcal{N}_n}{\sqrt{2\pi}} \frac{1}{E-z_n} \right) \tilde{\varphi}^+(E). \end{aligned} \quad (6.15)$$

The proofs of (6.12) and (6.14) are very similar to those of (6.7) and (6.10). In Eqs. (6.13) and (6.15), the integration is supposed to be done infinitesimally *above* the real axis of the second sheet, contrary to Eqs. (6.8) and (6.11), where the integration is supposed to be done infinitesimally *below* the real axis of the second sheet. Also, in Eqs. (6.13) and (6.15) the regulator is $e^{iE\alpha}$, $\alpha > 0$, whereas in Eqs. (6.8) and (6.11) the regulator is $e^{-iE\alpha}$, $\alpha > 0$. The reason why anti-resonances need the opposite sign in their regulator will become apparent in Sec. VIII.

Note that unlike $|z_n^- \rangle$ and $\langle +z_n|$, the “plus” Gamow ket $|z_n^+ \rangle$ and the “minus” Gamow bra $\langle -z_n|$ are not related to a Breit-Wigner amplitude in an obvious way.

The relation between the various representations we have constructed can be conveniently summarized in diagrams. For resonances we have

$$\begin{aligned} H; \varphi^-(r) \quad \Phi_{\text{exp}} &\subset L^2([0, \infty), dr) \subset \Phi_{\text{exp}}^\times \quad \langle r | z_n^- \rangle \equiv u(r; z_n) \\ &\downarrow U_- \qquad \qquad \downarrow U_- \qquad \qquad \downarrow U_- \\ \hat{H}; \hat{\varphi}^-(E) \quad \hat{\Phi}_{-\text{exp}} &\subset L^2([0, \infty), dE) \subset \hat{\Phi}_{-\text{exp}}^\times \quad \langle -E | z_n^- \rangle \equiv i\sqrt{2\pi}\mathcal{N}_n\delta(E - z_n) \\ &\downarrow \mathbb{A} \qquad \qquad \qquad \downarrow \mathbb{A} \\ \tilde{H}; \tilde{\varphi}^-(E) \quad \tilde{\Phi}_{-\text{exp}} &\subset L^2(\mathbb{R}, d_\alpha E) \subset \tilde{\Phi}_{-\text{exp}}^\times \quad \langle -E | z_n^- \rangle \equiv -\frac{\mathcal{N}_n}{\sqrt{2\pi}} \frac{1}{E-z_n} \end{aligned} \quad (6.16)$$

and

$$\begin{aligned} H; \varphi^+(r) \quad \Phi_{\text{exp}} &\subset L^2([0, \infty), dr) \subset \Phi_{\text{exp}}^\times \quad \langle +z_n | r \rangle \equiv u(r; z_n) \\ &\downarrow U_+ \qquad \qquad \downarrow U_+ \qquad \qquad \downarrow U_+ \\ \hat{H}; \hat{\varphi}^+(E) \quad \hat{\Phi}_{+\text{exp}} &\subset L^2([0, \infty), dE) \subset \hat{\Phi}_{+\text{exp}}^\times \quad \langle +z_n | E^+ \rangle \equiv i\sqrt{2\pi}\mathcal{N}_n\delta(E - z_n) \\ &\downarrow \mathbb{A} \qquad \qquad \qquad \downarrow \mathbb{A} \\ \tilde{H}; \tilde{\varphi}^+(E) \quad \tilde{\Phi}_{+\text{exp}} &\subset L^2(\mathbb{R}, d_\alpha E) \subset \tilde{\Phi}_{+\text{exp}}^\times \quad \langle +z_n | E^+ \rangle \equiv -\frac{\mathcal{N}_n}{\sqrt{2\pi}} \frac{1}{E-z_n} \end{aligned} \quad (6.17)$$

where \widehat{H} denotes the operator multiplication by E , $E \geq 0$, and \widetilde{H} denotes the operator multiplication by E , $-\infty < E < \infty$. The top, middle and bottom rows of these diagrams contain, respectively, the position, the energy and the $(-\infty, \infty)$ -“energy” representations. For anti-resonances, the diagrams are analogous.

VII. THE TIME EVOLUTION OF THE GAMOW STATES

We are now going to obtain the time evolution of the Gamow states by extending the time evolution operator $e^{-iHt/\hbar}$ into the spaces Φ'_{exp} and Φ^\times_{exp} . Since such extension was constructed in [95] to obtain the time evolution of the analytically continued Lippmann-Schwinger bras and kets, and since the Gamow states can be obtained from the analytically continued Lippmann-Schwinger bras and kets, the time evolution of the Gamow states will easily follow from the results of [95].

Let us calculate first the time evolution of the Gamow bra $\langle^+ z_n|$ for a resonance energy:

$$\begin{aligned} \langle^+ z_n|e^{-iHt/\hbar}|\varphi^+\rangle &= \langle^+ \widehat{z}_n|e^{-i\widehat{H}t/\hbar}|\widehat{\varphi}^+\rangle \\ &= \langle^+ \widehat{z}_n|e^{i\widehat{H}t/\hbar}|\widehat{\varphi}^+\rangle, \quad t < 0 \text{ only} \\ &= e^{iz_nt/\hbar}\widehat{\varphi}^+(z_n), \quad t < 0 \text{ only} \\ &= e^{iz_nt/\hbar}\langle^+ z_n|\varphi^+\rangle, \quad t < 0 \text{ only}, \quad \forall \widehat{\varphi}^+ \in \widehat{\Phi}_{+\text{exp}}; \end{aligned} \quad (7.1)$$

that is,

$$\langle^+ z_n|e^{-iHt/\hbar} = e^{iz_nt/\hbar}\langle^+ z_n|, \quad \text{only for } t < 0, \quad n = 1, 2, \dots \quad (7.2)$$

The reason why the time evolution for a Gamow bra associated with a resonant energy z_n is defined only for $t < 0$ is that when $t > 0$ and $z_n = E_n - i\Gamma_n/2$, the factor $e^{iz_nt/\hbar}$ blows up exponentially, and therefore $e^{i\widehat{H}t/\hbar}\widehat{\varphi}^+$ violates the bound (8.6) below. Hence, $e^{i\widehat{H}t/\hbar}\widehat{\varphi}^+$ is not in $\widehat{\Phi}_{+\text{exp}}$ when $t > 0$, and therefore the dual extension of $e^{i\widehat{H}t/\hbar}$ is not well defined [100]. We should also note that, strictly speaking, Eq. (7.1) does not prove that the time evolution of $\langle^+ z_n|$ is well defined for $t < 0$ in the sense of the theory of distributions. In order to prove so, one needs that the space of test functions Φ_{exp} be invariant under the action of $e^{iHt/\hbar}$ for $t < 0$. Since it is not known whether Φ_{exp} is invariant under the action of $e^{iHt/\hbar}$, it remains an open problem to show that Eq. (7.2) holds in the sense of the theory of distributions.

Let us now calculate the time evolution of the Gamow bra $\langle^+ z_n|$ for an anti-resonance

energy:

$$\begin{aligned}
\langle^+ z_n | e^{-iHt/\hbar} | \varphi^+ \rangle &= \langle^+ \widehat{z}_n | e^{-i\widehat{H}t/\hbar} | \widehat{\varphi}^+ \rangle \\
&= \langle^+ \widehat{z}_n | e^{i\widehat{H}t/\hbar} \widehat{\varphi}^\pm \rangle, \quad t > 0 \text{ only} \\
&= e^{iz_n t/\hbar} \widehat{\varphi}^+(z_n), \quad t > 0 \text{ only} \\
&= e^{iz_n t/\hbar} \langle^+ z_n | \varphi^+ \rangle, \quad t > 0 \text{ only}, \quad \forall \widehat{\varphi}^+ \in \widehat{\Phi}_{+\text{exp}}; \quad (7.3)
\end{aligned}$$

that is,

$$\langle^+ z_n | e^{-iHt/\hbar} = e^{iz_n t/\hbar} \langle^+ z_n |, \quad \text{only for } t > 0, \quad n = -1, -2, \dots \quad (7.4)$$

The reason why the time evolution for a Gamow bra associated with an anti-resonant energy $z_n = E_n + i\Gamma_n/2$ is defined only for $t > 0$ is that when $t < 0$, the factor $e^{iz_n t/\hbar}$ blows up exponentially, and therefore $e^{i\widehat{H}t/\hbar} \widehat{\varphi}^+$ violates the bound (8.6). Hence, $e^{iHt/\hbar} \widehat{\varphi}^+$ is not in $\widehat{\Phi}_{+\text{exp}}$ when $t < 0$, and therefore the dual extension of $e^{i\widehat{H}t/\hbar}$ is not well defined [100]. As in the case of Eq. (7.1), Eq. (7.3) does not prove that the time evolution of $\langle^+ z_n |$ is well defined for $t > 0$ in the sense of the theory of distributions when z_n is an anti-resonance energy.

The time evolution of the Gamow ket $|z_n^- \rangle$ associated with a resonant energy z_n is given by

$$\begin{aligned}
\langle \varphi^- | e^{-iHt/\hbar} | z_n^- \rangle &= \langle \widehat{\varphi}^- | e^{-i\widehat{H}t/\hbar} | \widehat{z}_n^- \rangle \\
&= \langle e^{i\widehat{H}t/\hbar} \widehat{\varphi}^- | \widehat{z}_n^- \rangle \\
&= (e^{iz_n^* t/\hbar} \widehat{\varphi}^-(z_n^*))^*, \quad t > 0 \text{ only} \\
&= e^{-iz_n t/\hbar} (\widehat{\varphi}^-(z_n^*))^*, \quad t > 0 \text{ only} \\
&= e^{-iz_n t/\hbar} \langle \varphi^- | z_n^- \rangle, \quad t > 0 \text{ only}, \quad \forall \widehat{\varphi}^+ \in \widehat{\Phi}_{+\text{exp}}; \quad (7.5)
\end{aligned}$$

that is,

$$e^{-iHt/\hbar} | z_n^- \rangle = e^{-iz_n t/\hbar} | z_n^- \rangle, \quad \text{only for } t > 0, \quad n = 1, 2, \dots \quad (7.6)$$

Similarly to Eqs. (7.2) and (7.4), Eq. (7.6) is clearly not defined for $t < 0$, although it remains to be proved that it holds for $t > 0$ in a distributional way.

When we consider an anti-resonance, it can be easily shown that Eq. (7.6) becomes

$$e^{-iHt/\hbar} | z_n^- \rangle = e^{-iz_n t/\hbar} | z_n^- \rangle, \quad \text{only for } t < 0, \quad n = -1, -2, \dots \quad (7.7)$$

Similarly to Eqs. (7.2), (7.4) and (7.6), Eq. (7.7) is clearly not defined for $t > 0$, although it remains to be proved that it holds for $t < 0$ in a distributional way.

In summary, the time evolution of the Gamow states is given by non-unitary semigroups and therefore is time asymmetric, expressing the irreversibility of a decaying process. Such semigroups are simply (retarded or advanced) propagators that incorporate causal boundary conditions through the analytical properties of the test functions [95]. However, as explained above, the rigorous proof of (7.2), (7.4), (7.6) and (7.7) is still lacking, because it is not known whether Φ_{exp} is invariant under $e^{-iHt/\hbar}$.

VIII. RESONANCE EXPANSIONS

The Lippmann-Schwinger bras and kets are basis vectors that were used to expand normalizable, smooth wave functions in [94]:

$$\langle r|\varphi^\pm\rangle = \int_0^\infty dE \langle r|E^\pm\rangle \langle^\pm E|\varphi^\pm\rangle. \quad (8.1)$$

The Gamow states are also basis vectors. The expansion generated by the Gamow states is called the resonance expansion.

A given quantity (wave function, amplitude, etc.) can be expanded by resonance states in many different ways, depending on how many resonances we include in the expansion, see e.g. review [44]. When we include only a few resonances close to the real axis, as in Berggren's and Berggren-like resonance expansions, the wave functions $\varphi(r)$ must fall off at infinity faster than exponentials [23, 24]. However, when we include all the resonances, we will see that the wave functions must fall off faster than Gaussians.

For the sake of simplicity, we will focus on the resonance expansion of the transition amplitude from an “in” state φ^+ into an “out” state φ^- :

$$(\varphi^-, \varphi^+) = \int_0^\infty dE \langle \varphi^-|E^-\rangle S(E) \langle^+ E|\varphi^+\rangle, \quad (8.2)$$

where $S(E)$ is the S matrix. For the spherical shell potential, and also for any spherically symmetric potential that falls off faster than exponentials, the S -matrix and the Lippmann-Schwinger eigenfunctions can be analytically continued to the whole complex plane (see Appendix A of Ref. [63], and references therein). Thus, by using the contour of Fig. 1, we obtain

$$(\varphi^-, \varphi^+) = \sum_{n=1}^{\infty} \langle \varphi^-|z_n^-\rangle \langle^+ z_n|\varphi^+\rangle + \int_0^{-\infty} dE \langle \varphi^-|E^-\rangle S(E) \langle^+ E|\varphi^+\rangle, \quad (8.3)$$

where we have tacitly assumed that $\langle \varphi^- | E^- \rangle S(E) \langle +E | \varphi^+ \rangle$ tends to zero in the infinite arc of the lower half plane of the second sheet. The integral in Eq. (8.3) is done infinitesimally below the negative real semiaxis of the second sheet. By omitting φ^- in (8.3), we obtain the resonance expansion of the “in” wave functions,

$$\varphi^+ = \sum_{n=1}^{\infty} |z_n^- \rangle \langle +z_n | \varphi^+ \rangle + \int_0^{-\infty} dE |E^- \rangle S(E) \langle +E | \varphi^+ \rangle. \quad (8.4)$$

The resonance expansion for the “out” wave function φ^- can be obtained in a similar way. In Eqs. (8.3) and (8.4), the infinite sum exhibits explicitly the contribution from the resonances, while the integral is the non-resonant background.

In obtaining Eqs. (8.3) and (8.4), we have tacitly assumed that $\langle \varphi^- | E^- \rangle S(E) \langle +E | \varphi^+ \rangle$ tends to zero in the infinite arc of the lower half plane of the second sheet. However, as shown in [95], $\langle \varphi^- | E^- \rangle S(E) \langle +E | \varphi^+ \rangle$ diverges exponentially there, since for any $\beta > 0$ there is a constant C such that [95]

$$|\langle \varphi^- | z^- \rangle| \leq C |q|^{-1/2} e^{\frac{|\text{Im}(q)|^2}{2\beta}}, \quad (8.5)$$

$$|\langle +z | \varphi^+ \rangle| \leq C |q|^{-1/2} e^{\frac{|\text{Im}(q)|^2}{2\beta}}, \quad (8.6)$$

where q is the corresponding complex wave number in the fourth quadrant of the k -plane. Therefore, Eqs. (8.3) and (8.4) need to be established properly. In order to do so, one has to control the exponential blowups (8.5) and (8.6) by calculating the time evolution of Eqs. (8.3) and (8.4):

$$(\varphi^-, e^{-iHt/\hbar} \varphi^+) = \sum_{n=1}^{\infty} e^{-iz_n t/\hbar} \langle \varphi^- | z_n^- \rangle \langle +z_n | \varphi^+ \rangle + \int_0^{-\infty} dE e^{-iEt/\hbar} \langle \varphi^- | E^- \rangle S(E) \langle +E | \varphi^+ \rangle, \quad t > 0 \text{ only}, \quad (8.7)$$

$$e^{-iHt/\hbar} \varphi^+ = \sum_{n=1}^{\infty} e^{-iz_n t/\hbar} |z_n^- \rangle \langle +z_n | \varphi^+ \rangle + \int_0^{-\infty} dE e^{-iEt/\hbar} |E^- \rangle S(E) \langle +E | \varphi^+ \rangle, \quad t > 0 \text{ only}. \quad (8.8)$$

These equations are valid because the following limits hold in the infinite arc of the lower half plane of the second sheet for any $\alpha > 0$:

$$\lim_{z \rightarrow \infty} e^{-i\alpha z} \langle \varphi^- | z^- \rangle = \lim_{z \rightarrow \infty} e^{-i\alpha z} \langle +z | \varphi^+ \rangle = 0, \quad (8.9)$$

which in turn follow from Eqs. (8.5) and (8.6) (see however [100]). Equations (8.3) and (8.4) should then be understood as the limit of Eqs. (8.7) and (8.8) when $t \rightarrow 0^+$.

As shown in [23, 24], the Gamow bras and kets are already well defined when the tails of the test functions fall off like exponentials rather than like Gaussians. The reason why we chose a Gaussian falloff has finally become clear. For test functions with exponential falloff, the above resonance expansions make no sense, since there is no way we can regularize the blowup of such test functions in the infinite arc of the second sheet [101]. However, imposing a Gaussian falloff on the elements of Φ_{exp} enables us to regularize their blowup in the complex energy plane by using the time evolution phase $e^{-izt/\hbar}$ as a regulator. Also, it is clear that Gaussian falloff is the slowest falloff that can be regularized in this way.

As is well known, resonance expansions allow us to understand the deviations from exponential decay. If a particular resonance, say resonance 1, is dominant, then Eq. (8.8) can be written as

$$e^{-iHt/\hbar}\varphi^+ = e^{-iz_1t/\hbar}|z_1^-\rangle\langle^+z_1|\varphi^+\rangle + \text{background}(1), \quad (8.10)$$

where the term “background(1)” carries the contributions not associated with resonance 1, including those from other resonances. Because “background(1)” never vanishes, there are always deviations from exponential decay. The exponential law holds only when the wave function is well tuned around the Gamow state $|z_1^-\rangle$, in which case “background(1)” can be neglected and only the resonance (Gamow state) contribution to the probability needs to be taken into account.

IX. PHYSICAL MEANING OF THE GAMOW STATES

We are now going to explore the physical meaning of the Gamow states. We will do so by way of two analogies. The first analogy is that between classical Fourier expansions, quantum completeness relations and resonance expansions. The second analogy is that between the Gamow states and the quasinormal modes of classical systems. As always when one draws analogies between classical and quantum mechanics, one should keep in mind that in classical mechanics the solutions of the wave equations are actual waves, whereas in quantum mechanics the solutions of the Schrödinger equation are probability amplitudes.

A. Plane waves, the Lippmann-Schwinger bras and kets, and the Gamow states

Plane waves e^{ikx} represent monochromatic light pulses of well-defined wave number k . Experimentally, one cannot prepare monochromatic plane waves: all that one can prepare are wave packets $\widehat{\varphi}(k)$ that have some wave-number spread. The corresponding wave packet in the position representation, $\varphi(x)$, can be expanded in terms of the plane waves as

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \widehat{\varphi}(k), \quad (9.1)$$

which in Dirac's notation is written as

$$\langle x|\varphi\rangle = \int dk \langle x|k\rangle \langle k|\varphi\rangle. \quad (9.2)$$

When $\widehat{\varphi}(k)$ is highly peaked around a particular wave number k_0 , the wave packet is well approximated by a monochromatic plane wave, $\varphi(x) \sim e^{ik_0x}$.

The Lippmann-Schwinger bras and kets are a quantum version of the classical plane waves. The monoenergetic eigenfunctions $\langle r|E^\pm\rangle$ represent a particle with a sharply defined energy E (and with additional “in” or “out” boundary conditions). In analogy to the Fourier expansion of wave packets in terms of classical plane waves, Eq. (9.1), the eigenfunctions $\langle r|E^\pm\rangle$ expand wave functions φ^\pm as in Eq. (8.1). When the wave packet $\widehat{\varphi}^\pm(E)$ is highly peaked around a particular energy E_0 , then the approximation $\varphi^\pm(r) \sim \langle r|E_0^\pm\rangle$ holds.

The physical meaning of the Gamow states is similar. Likewise the monoenergetic scattering states, the Gamow states cannot be prepared experimentally: All that can be prepared is a wave packet φ^+ . In complete analogy to the expansions (9.1) and (8.1), the Gamow states and an additional set of “background” states expand a wave function φ^+ , see Eq. (8.4). When the wave function is finely tuned around one resonance, say resonance 1, then in general the approximation $\varphi^+(r) \sim \langle r|z_1\rangle$ holds for all practical purposes [102]. It is in this sense that a lone Gamow state is the wave function of a quantum decaying particle.

When the approximation $\varphi^+(r) \sim \langle r|z_1\rangle$ holds, the Gamow state can be used to characterize the transport of probability in a time-dependent description of resonant scattering and decay. For instance, in Ref. [50] Garcia-Calderon *et al.* present the example of a delta-shell potential where one resonance dominates the decay of the system. In order to show so, the authors of [50] calculate the survival probability using a square integrable function φ^+ . They also use a resonant expansion to approximate φ^+ by one single resonant state $\langle r|z_1\rangle$.

As shown in Fig. 3 of Ref. [50], the exponential decay of the survival probability calculated by way of the Gamow state $\langle r|z_1\rangle$ is indistinguishable from the one calculated by way of the “exact” square integrable wave function φ^+ .

B. Quasinormal modes vs. resonance states

In classical mechanics, confined linear oscillating systems –e.g., finite strings, membranes or cavities filled with electromagnetic radiation– have preferred states of motion. Such states of motion are called normal modes. Each normal mode is associated with a characteristic real frequency. Unless it is perturbed, a system in a normal mode will keep vibrating the same way perpetually. When friction or dissipation enters into play and therefore the system dissipates energy, the system has preferred ways of doing so, which are called the quasinormal modes. Unconfined linear oscillating systems also have quasinormal modes, and they are obtained by imposing Sommerfeld’s radiation condition, which is the classical counterpart of the POBC [103]. Each quasinormal mode is associated with a characteristic complex frequency, whose imaginary part is associated with the exponential damping of the oscillation.

In quantum mechanics, normal modes correspond to bound states, and quasinormal modes correspond to resonance states. Much like quasinormal modes describe the system’s preferred ways of dissipating energy, the Gamow states describe the system’s preferred ways of decaying. The imaginary part of the complex energy of the Gamow state is associated with exponential decay, in analogy to the imaginary part of the complex, classical frequency being associated with exponential dissipation.

C. Physical meaning of the exponential blowup of the Gamow states

The Gamow states blow up exponentially at infinity, and it is important to understand the physical origin of such exponential blowup. Let us consider first the Lippmann-Schwinger eigenfunction $\langle r|E^+\rangle$. These time-independent, non-normalizable eigenfunctions are interpreted as an incoming plane wave that impinges on a target and an outgoing wave multiplied by the S matrix. However, the actual expression of $\langle r|E^+\rangle$ does not lead to such interpretation. Only when one views the Lippmann-Schwinger eigenfunction in a time-dependent

fashion, one can arrive at such interpretation. Thus, even though they are time-independent, the Lippmann-Schwinger eigenfunctions encode what happens in a scattering experiment at all times.

Similarly, the Gamow states are time-independent, non-normalizable eigenfunctions that describe the decay of a quantum system at all times. Since after a long time (formally, when $t \rightarrow \infty$) the resonance will surely have decayed and gone to infinity, the Gamow state needs to provide a time-independent probability amplitude of finding the particle at infinity that is much greater than the probability of finding the particle anywhere else in space, hence the exponential blowup at infinity.

X. CONCLUSIONS

We have used the spherical shell potential to present a systematic procedure to construct the rigged Hilbert space of the Gamow states. A Gamow state has been defined as the solution of the homogeneous integral equation introduced in [11, 14]. Such integral equation is of the Lippmann-Schwinger type, and is equivalent to the Schrödinger equation subject to the POBC.

By applying the theory of distributions, we have constructed the Gamow bras and kets and shown that they are, respectively, linear and antilinear functionals over the space of test functions Φ_{exp} , where the elements of Φ_{exp} are smooth functions that fall off faster than Gaussians. We have shown that the Gamow bras and kets are, respectively, “left” and “right” eigenvectors of the Hamiltonian, and that their associated eigenvalues coincide with the resonance energies. We have argued, although not rigorously proved, that the exponential time evolution is given by a non-unitary semigroup. Such semigroup time evolution exhibits the time asymmetry of a decaying process.

We have also constructed the energy representations of the Gamow states. We have shown that such energy representations are given by either the complex delta function or by the residue distribution. These results complement the properties of the Gamow states in the momentum representation obtained in [14, 23, 24].

Because in the position representation the wave functions in Φ_{exp} fall off faster than Gaussians, we have been able to construct resonance expansions that include all the resonances. Such resonance expansions also exhibit the time asymmetry of the decaying process.

Finally, we have clarified some of the physical properties of the Gamow states by drawing analogies with classical Fourier expansions and quasinormal modes. We have also clarified the origin of the exponential blowup of a Gamow state at infinity.

XI. ACKNOWLEDGMENT

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Appendix A: Proofs

Here we list the proofs of some results we stated in the paper. In the proofs, whenever an operator A is acting on the bras, we will use the notation A' , and whenever it is acting on the kets, we will use the notation A^\times :

$$\langle^\pm z_n | A' | \varphi^\pm \rangle := \langle^\pm z_n | A^\dagger \varphi^\pm \rangle, \quad \forall \varphi^\pm \in \Phi_{\text{exp}}, \quad (\text{A1})$$

$$\langle \varphi^\pm | A^\times | z_n^\pm \rangle := \langle A^\dagger \varphi^\pm | z_n^\pm \rangle, \quad \forall \varphi^\pm \in \Phi_{\text{exp}}. \quad (\text{A2})$$

Thus, A' denotes the *dual* extension of A acting to the left on the elements of Φ'_{exp} , whereas A^\times denotes the *antidual* extension of A acting to the right on the elements of Φ^\times_{exp} . This notation stresses that A is acting outside the Hilbert space and specifies toward what direction the operator is acting, thereby making the proofs more transparent.

Proof of Proposition 2.

The proofs of Eqs. (5.14)-(5.17) all follow the same pattern. We start by proving (5.15). The Gamow eigenfunction $u(r; z_n)$ is proportional to the analytic continuation of the Lippmann-Schwinger eigenfunction $\chi^-(r; E)$ [55],

$$u(r; z_n) = i\sqrt{2\pi} \mathcal{N}_n \chi^-(r; z_n). \quad (\text{A3})$$

From this equation and from the analytic properties of the elements $\widehat{\varphi}^- \in \Phi_{-\text{exp}}$ obtained in [95], it follows that

$$\langle \widehat{\varphi}^- | \widehat{z}_n^- \rangle = \langle \widehat{\varphi}^- | U_-^\times | z_n^- \rangle$$

$$\begin{aligned}
&= \langle U_-^\dagger \widehat{\varphi}^- | z_n^- \rangle \\
&= \langle \varphi^- | z_n^- \rangle \\
&= \int_0^\infty dr [\varphi^-(r)]^* u(r; z_n) \quad \text{by (3.1)} \\
&= i\sqrt{2\pi} \mathcal{N}_n \int_0^\infty dr [\varphi^-(r)]^* \chi^-(r; z_n) \quad \text{by (A3)} \\
&= i\sqrt{2\pi} \mathcal{N}_n [\widehat{\varphi}^-(z_n^*)]^* \\
&= i\sqrt{2\pi} \mathcal{N}_n \langle \widehat{\varphi}^- | \widehat{\delta}_{z_n} \rangle, \quad \forall \widehat{\varphi}^- \in \widehat{\Phi}_{-\text{exp}}, \tag{A4}
\end{aligned}$$

which proves (5.15). The proof of (5.16) is analogous.

In order to prove (5.17), we need the following relation [55]:

$$u(r; z_n) = -\frac{\sqrt{2\pi}}{\mathcal{N}_n} \text{res} [\chi^+(r; z)]_{z=z_n}. \tag{A5}$$

Then,

$$\begin{aligned}
\langle \widehat{\varphi}^+ | \widehat{z}_n^+ \rangle &= \langle \widehat{\varphi}^+ | U_+^\times | z_n^+ \rangle \\
&= \langle U_+^\dagger \widehat{\varphi}^+ | z_n^+ \rangle \\
&= \langle \varphi^+ | z_n^+ \rangle \\
&= \int_0^\infty dr [\varphi^+(r)]^* u(r; z_n) \quad \text{by (3.1)} \\
&= -\frac{\sqrt{2\pi}}{\mathcal{N}_n} \int_0^\infty dr [\varphi^+(r)]^* \text{res} [\chi^+(r; z)]_{z=z_n} \quad \text{by (A5)} \\
&= -\frac{\sqrt{2\pi}}{\mathcal{N}_n} \text{res} [\widehat{\varphi}^+(z_n^*)]^* \\
&= -\frac{\sqrt{2\pi}}{\mathcal{N}_n} \langle \widehat{\varphi}^+ | \widehat{\text{res}}_{z_n} \rangle, \quad \forall \widehat{\varphi}^+ \in \widehat{\Phi}_{+\text{exp}}, \tag{A6}
\end{aligned}$$

which proves (5.17). The proof of (5.14) is analogous.

Proof of Eq. (6.7).

Let $\widehat{\varphi}^- \in \widehat{\Phi}_{-\text{exp}}$. It was proved in [95] that for any $\beta > 0$, the following estimate is valid in the lower half plane of the second sheet:

$$|[\widehat{\varphi}^-(z^*)]^*| \leq C|q|^{-1/2} e^{\frac{|\text{Im}(q)|^2}{2\beta}}, \tag{A7}$$

where q is the corresponding complex wave number in the fourth quadrant of the k -plane. This estimate implies that in the infinite arc of the lower half plane of the second sheet, the

following limit holds for any $\alpha > 0$ (see however [100]):

$$\lim_{|z| \rightarrow \infty} e^{-i\alpha z} [\widehat{\varphi}^-(z^*)]^* = 0. \quad (\text{A8})$$

Then, by Cauchy's formula,

$$e^{-i\alpha z_n} [\widehat{\varphi}^-(z_n^*)]^* = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dE e^{-i\alpha E} \frac{1}{E - z_n} [\widehat{\varphi}^-(E)]^*. \quad (\text{A9})$$

Multiplying this equation by $i\sqrt{2\pi}\mathcal{N}_n$ yields

$$e^{-i\alpha z_n} i\sqrt{2\pi}\mathcal{N}_n [\widehat{\varphi}^-(z_n^*)]^* = \int_{-\infty}^{\infty} dE e^{-i\alpha E} \left(-\frac{\mathcal{N}_n}{\sqrt{2\pi}} \right) \frac{1}{E - z_n} [\widehat{\varphi}^-(E)]^*. \quad (\text{A10})$$

From Eqs. (6.6), (6.8) and (A10) it follows that

$$i\sqrt{2\pi}\mathcal{N}_n [\widetilde{\varphi}^-(z_n^*)]^* = \langle \widetilde{\varphi}^- | \frac{1}{E - z_n}^- \rangle. \quad (\text{A11})$$

We now define the action of \mathbb{A}^\times on $|\widehat{z}_n^-\rangle$ by

$$\langle \widetilde{\varphi}^- | \mathbb{A}^\times | \widehat{z}_n^- \rangle := \langle \mathbb{A}^{-1} \widetilde{\varphi}^- | \widehat{z}_n^- \rangle. \quad (\text{A12})$$

Since

$$\langle \mathbb{A}^{-1} \widetilde{\varphi}^- | \widehat{z}_n^- \rangle = \langle \widehat{\varphi}^- | \widehat{z}_n^- \rangle = i\sqrt{2\pi}\mathcal{N}_n [\widehat{\varphi}^-(z_n^*)]^* = i\sqrt{2\pi}\mathcal{N}_n [\widetilde{\varphi}^-(z_n^*)]^*, \quad (\text{A13})$$

we have that

$$\langle \widetilde{\varphi}^- | \mathbb{A}^\times | \widehat{z}_n^- \rangle = \langle \widetilde{\varphi}^- | \frac{1}{E - z_n}^- \rangle, \quad \forall \widetilde{\varphi}^- \in \widetilde{\Phi}_{\text{exp}}, \quad (\text{A14})$$

which proves (6.7).

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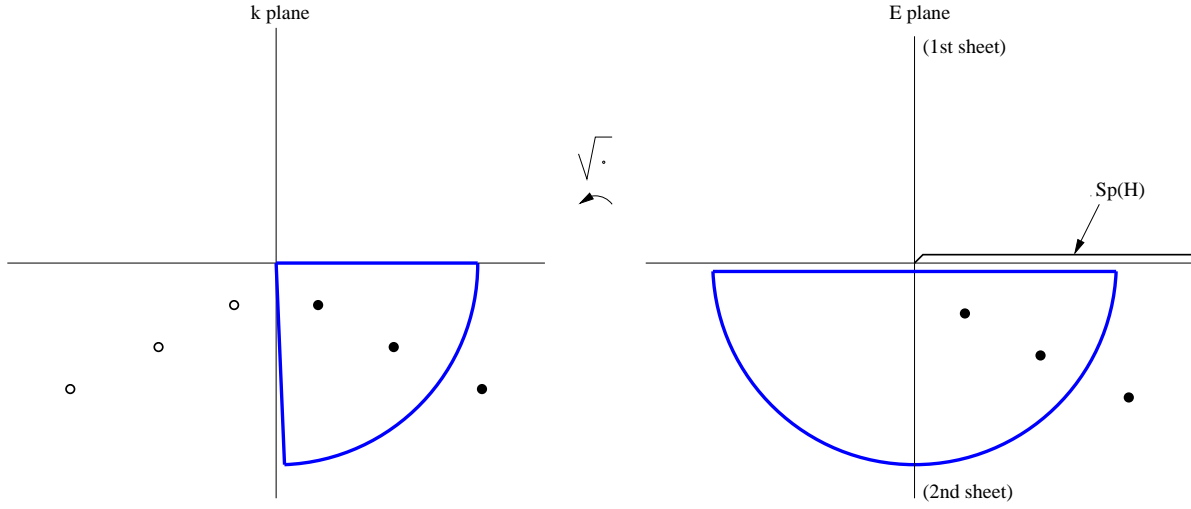


FIG. 1: The contour to obtain resonance expansions in the k -plane (left) and in the E -plane (right). It is assumed that the contour encloses all the resonances in the lower half plane of the second sheet, and that the radius of the arc is sent to infinity. The filled (hollow) dots represent the resonance (anti-resonance) poles.